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COMMENT

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Abstract. The validity of the simple Green function used in wave propagation problems is established by making use of the mathematically exact one.

In a recent paper Marathay (1975) has pointed out that the Green function (equation (2.9) in Byckling 1974, to be referred to as I) used by us to calculate wave propagation in optical systems is not identical with the mathematically correct one. Marathay, however, makes the conjecture that it probably gives correct results in all practical calculations. Using the exact Green function we show that, in fact, all the significant conclusions of the paper are valid.

The situation and symbols are those of figure 1 and chapter 2 in I. For the problem, Marathay has constructed a Green function, which satisfies the following conditions:

$$(\nabla'^2+n'^2k^2)G(\boldsymbol{x},\boldsymbol{x}')=-\delta(\boldsymbol{x}-\boldsymbol{x}'),$$
$$x>x'\Rightarrow G(\boldsymbol{x},\boldsymbol{x}')|_{x'=r}=0$$

which imply

$$A(\boldsymbol{t}')=-\int_{r'=r}A(\boldsymbol{r}')\frac{\partial G(\boldsymbol{t}',\boldsymbol{r}')}{\partial r'}r'^2\,\mathrm{d}\Omega. \tag{1}$$

Moreover one has

$$\left.\frac{\partial G(\boldsymbol{t}',\boldsymbol{r}')}{\partial r'}\right|_{r'=r}=-\frac{1}{r^2}\sum_{l=0}^{\infty}\frac{h_l^{(1)}(n'kt')}{h_l^{(1)}(n'kr)}\frac{2l+1}{4\pi}P_l(\cos\theta(\boldsymbol{r},\boldsymbol{t}')).$$

Substituting

$$A(\boldsymbol{r})=\frac{\exp(\mathrm{i}nk|\boldsymbol{r}-\boldsymbol{t}|)}{|\boldsymbol{r}-\boldsymbol{t}|}=\mathrm{i}nk\sum_{l=0}^{\infty}j_l(nkr)h_l^{(1)}(nkt)(2l+1)P_l(-\cos\theta)$$

in (1) we get, using the result of integration given by equation (A.7) in I:

$$A(t')=\frac{2\mathrm{i}nk}{\pi}\sum_{l,l'}(ll')^{1/2}\frac{\sin(l-l')\hat{\theta}}{(l-l')}\mathrm{e}^{\pm\mathrm{i}\pi l}\frac{h_{l'}^{(1)}(n'kt')}{h_{l'}^{(1)}(n'kr)}j_l(nkr)h_l^{(1)}(nkt)$$

where we have used $P_l(-\cos\theta)=(-1)^lP_l(\cos\theta)=\mathrm{e}^{\pm\mathrm{i}\pi l}P_l(\cos\theta)$, and $(2l+1)(2l'+1)\cong 4ll'$. Now using the asymptotic formula for $h_l^{(1)}$ given in I, equation (A.14),

$$h_l^{(1)}(z)\underset{z\to\infty}{\longrightarrow}z^{-1/2}(z^2-l^2)^{-1/4}\exp\{\mathrm{i}[(z^2-l^2)^{1/2}-l\cos^{-1}(l/z)-\tfrac{1}{4}\pi]\}$$

we get

$$A(t') = \frac{2i}{\pi(tt')^{1/2}} \sum_{l,l'} \left[(ll')^{1/2} e^{\pm i\pi l} \frac{\sin(l-l')\hat{\theta}}{(l-l')} \left((n^2k^2r^2 - l^2) \times \frac{(n'^2k^2t'^2 - l'^2)}{(n'^2k^2r^2 - l'^2)} (n^2k^2t^2 - l^2) \right)^{-1/4} \cdot \cos \Phi_l \exp(-i\Phi'_{l'} + i\Psi_l + i\Psi'_{l'}) \right]. \quad (2)$$

Here the functions Φ_l, Ψ_l are as in equations (2.17) and (2.18) of I. If we compare (2) with equation (2.16) of I, we note the following differences:

(i) If $n'kr \gg l'$, the new modulus of $A(t')$ is $(n'k\pi)^{-1}$ times the old one.

(ii) The old and new phases are:

$$\phi_{ll'}^{\text{old}} = \pm\Phi_l \pm \Phi'_{l'} - \Psi_l - \Psi'_{l'} \pm \pi l \quad (3)$$

$$\phi_{ll'}^{\text{new}} = \pm\Phi_l - \Phi'_{l'} + \Psi_l + \Psi'_{l'} \pm \pi l. \quad (4)$$

The calculations made in I were based on the method of stationary phase which requires $\partial\phi_{ll'}/\partial l = 0$. The correct sign combinations in (3) were then shown to be

$$\phi_{ll'}^{\text{old}} = -\Phi_l + \Phi'_{l'} - \Psi_l - \Psi'_{l'} - \pi l.$$

From (4) it is seen that the result

$$\phi_{ll'}^{\text{old}} = -\phi_{ll'}^{\text{new}}$$

can be obtained by choosing appropriate signs in (4). The sign change arises from the trivial difference in the definitions in the two papers. Thus the wave-tracing calculations in I remain the same whether we use the Green function of I or the exact one of Marathay.

References

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